On open 3-manifolds proper homotopy equivalent to geometrically simply-connected polyhedra *

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Abstract

We prove that an open 3-manifold proper homotopy equivalent to a geometrically simply connected polyhedron is simply connected at infinity thereby generalizing the theorem proved by Poénaru in [6].

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1 Introduction

The immediate antecedent to this paper is [6], the principal theorem of which is the following.

Theorem 1.1 *(V.Poénaru) If U is an open simply connected 3-manifold and, for some n, $U \times D^n$ has a handlebody decomposition without 1-handles then U is geometrically simply connected hence simply connected at infinity.*

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Note: D^n denotes the n-ball; see [6] for the definition of "handlebody decomposition without 1-handles "; all 3-manifolds we consider in the sequel will be orientable unless the contrary is explicitly stated.

A non-compact polyhedron U is *simply connected at infinity* (s.c.i.), and we write also $\pi_1^\infty(U) = 0$, if given a compactum (i.e. a compact set) $X \subset U$ there exists another compactum Y with $X \subset Y \subset U^3$, such that any loop in $U - Y$ is null-homotopic in $U - X$. (Some authors call this π_1-*triviality at infinity* or *1-LC at infinity* and reserve the term s.c.i. for the special case in which Y can be chosen so that, in addition, $U - Y$ is connected. These notions are equivalent for one ended spaces, such as contractible spaces.

A non-compact polyhedron P is *geometrically simply connected (g.s.c.)* if it can be exhausted by compact 1-connected subpolyhedra. It is easily seen that any non-compact polyhedral manifold U which has a handlebody decomposition with no 1-handles is g.s.c. In addition the projection map $p : U \times D^n \to U$ is a proper simple-homotopy equivalence (defined in [7]). In [6] Poénaru hinted at the conjecture which results when the hypothesis of the above-stated theorem is replaced by the (therefore) weaker hypothesis that U be proper simple-homotopy equivalent to a g.s.c. polyhedron. This conjecture was subsequently established in [2] using the techniques of [6]. The following theorem is an immediate corollary of the principal result of this paper (which is proven using only basic 3-manifold theory). It further generalizes the theorem stated above.

Theorem 1.2 *Any open 3-manifold which is proper homotopy equivalent to a geometrically simply connected polyhedron is simply-connected at infinity.*

Remark 1.1 *If M^n $(n > 3)$ is a compact, contractible n-manifold with non-simply-connected boundary (e.g. those constructed in [4] and [5]) then int(M^n) is easily seen to be g.s.c. but not s.c.i. This demonstrates that the above theorem cannot be extended to include open contractible manifolds of dimension greater than three. Notice that for simply connected open 3-manifolds the g.s.c. and the s.c.i. are equivalent.*

Provisos: We remain in the polyhedral category throughout and all homology groups are with $\mathbf{Z}$ coefficients.

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Several months after the appearance of [2] in preprint form, the second-named author of this paper communicated to the first-named author an outline of the proof of the principal result found here (the details were then worked out jointly).

2 Statement of results

We first require the following definitions.

Definition 2.1 A proper map $f : X \to Y$ is *H_3-nontrivial* if given non-null compacta $L \subset Y$ and $K \subset X$ such that $f(X-K) \subset Y-L$ then $f_* : H_3(X, X-K) \to H_3(Y, Y-L)$ is nontrivial (i.e. its image is not a singleton).

Definition 2.2 Given noncompact polyhedra X and Y we say Y *is H_3-semi-dominated by* X if there exists an H_3-nontrivial proper map $f : X \to Y$.

Definition 2.3 An open connected 3-manifold U^3 is *simple-ended* if it has an exhaustion $\{M_i\}_{i=1}^{\infty}$ by compact 3-submanifolds where, for all i, the genus of ∂M_i is zero.

Remark 2.1
1. *If U is an open orientable 3-manifold and $Y \subset U$ a non-null compactum then $H_3(U, U-Y)$ is nontrivial.*
2. *If $f, g : X \to Y$ are properly homotopic maps where X and Y are open, connected 3-manifolds and f is H_3-nontrivial then g is H_3-nontrivial.*

Proof of 1.: Since we are in the polyhedral category Y is a compact polyhedron and so has a regular neighborhood N. Since $N - Y$ deformation retracts onto ∂N we have that $H_3(U, U-Y)$ is isomorphic to $H_3(U, U-int\ N)$. Taking a triangulation of the pair (U, N) and using the orientation of U we see that N is a relative cycle representing a nontrivial element of $H_3(U, U - int\ N)$. Moreover, this is a free abelian group with basis elements represented by the components of N. This proves the first remark.

Proof of 2.: Suppose that K, L are compacta in X, Y, respectively, with $g(X-K) \subset Y-L$. Then $K \supset g^{-1}(L)$. Let $F : X \times I \to Y$ be a proper homotopy with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Since $F^{-1}(L)$ is compact, there is a connected compactum M of X such that $M \times I \supset F^{-1}(L) \cup (K \times \{1\})$. Let N be a regular neighborhood of M in X. Then N represents a generator of the infinite cyclic group $H_3(X, X - int\,N)$ which is isomorphic to $H_3(X, X-M)$. Now $f^{-1}(L) \subset M$ implies that $f(X-M) \subset Y-L$. Since f is H_3-nontrivial we have that $f(N)$ represents a nontrivial element of $H_3(Y, Y-L)$. Since $K \subset int\,N$ we have that N represents an element of $H_3(X, X-K)$. Since $F^{-1}(L) \subset int\,N \times I$ we have that $g(N)$ is homologous to $f(N)$ in $(Y, Y-L)$, and so g is H_3-nontrivial. $\square$

Our main result is the following.

Theorem 2.1 *An open, connected, orientable 3-manifold U is H_3-semi-dominated by a g.s.c. polyhedron if and only if U is the connect-sum of a 1-connected, simply connected at infinity open 3-manifold and a closed, orientable 3-manifold with finite fundamental group.*

Remark 2.2 *1. Theorem 1.2. is a corollary of Theorem 2.1. In fact, let $f : P \to U$ be a proper homotopy equivalence with proper homotopy inverse $g : U \to P$. Suppose K, L are compacta in P, U, respectively, such that $f(P - K) \subset U - L$. Let $M = g^{-1}(K)$. Then $g(U - M) \subset P - K$. Since $f \circ g$ is proper homotopic to id_U and id_U is H_3-nontrivial, we have by Remark 2.1 (2) that $(f \circ g)_*$ is nontrivial and hence f_* is nontrivial.*

2. *The proof of the "if" part of the theorem is very brief. Just observe that if U satisfies the hypothesis then the universal covering of U is 1-connected and simply connected at infinity (hence g.s.c. -see (3) below) and the covering projection is proper and has non-zero degree (hence is H_3-nontrivial). (In the sequel when we refer to the hypothesis or conclusion of the theorem we will mean the "only if" part.)*

3. *By the methods of [9] the class of 1-connected, simply connected at infinity open 3-manifolds is equal to each of the following two classes of open 3-manifolds. Those which can be constructed as follows: delete a tame, 0-dimensional, compact subspace from S^3, denote the result by U and replace each element of a pairwise disjoint, proper family of 3-balls in U by a homotopy 3-ball. Those each of which has an exhaustion $\{M_i\}_{i=1}^{\infty}$ by compact 3-submanifolds where, for each i, M_i is 1-connected (and hence the genus of ∂M_i is 0) and each component of $M_{i+1} - M_i$ is homeomorphic to a space obtained by taking finitely many pairwise disjoint 3-balls in S^3, replacing one by a homotopy 3-ball and deleting the interiors of the rest.*

To establish the theorem we will demonstrate the following three propositions.

Proposition 2.1 *If U is as in the hypothesis of the theorem then U is simple-ended.*

Proposition 2.2 *If U is as in the hypothesis of the theorem then $\pi_1(U)$ is a torsion group.*

Proposition 2.3 *If U is an open, connected simple-ended 3-manifold such that $\pi_1(U)$ is a torsion group then U is as in the conclusion of the theorem.*

3 Proof of Proposition 2.1

Lemma 3.1 *Suppose the following: U is an open, orientable 3-manifold; K is a compact, connected 3-submanifold of U such that each component of $cl(U - K)$ is noncompact and has connected boundary; and $f : (M, \partial M) \to (U, U - K)$ is a map of a compact connected 3-manifold with boundary such that ∂M has genus zero and $f_* : H_3(M, \partial M) \to H_3(U, U - K)$ is nontrivial. Then there exists a compact, connected 3-submanifold N of U such that K is in N and the genus of ∂N is zero.*

Proof: It will suffice to find, for each component V of $cl(U-K)$ a compact, connected 3-submanifold $N(V)$ of V such that ∂V is in $N(V)$ and the genus of $\partial N(V) - \partial V$ is zero. So let V be such a component. We assume f is transverse to ∂V and denote the intersection of $f^{-1}(V)$ and ∂M by C. The orientation of U determines an orientation of V, which in turn determines an orientation of ∂V. Similarly the orientation of M determines an orientation of the 3-dimensional submanifold $f^{-1}(V)$, which in turn determines an orientation of $f^{-1}(\partial V) \subset \partial f^{-1}(V)$.

From the hypothesis on f_* we derive that $(f|_{f^{-1}(\partial V)})_*$ carries the homology class of $f^{-1}(\partial V)$, as determined by the orientation above, to a nontrivial element of $H_2(\partial V)$. In fact, consider the commutative diagram

$$\begin{array}{ccc} H_3(M,\partial M) & \xrightarrow{\partial_M} & H_2(f^{-1}(\partial V)) \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle (f|_{f^{-1}(\partial V)})_*} \\ H_3(U, U - int\, K) & \xrightarrow{\partial_U} & H_2(\partial V) \end{array}$$

where ∂_M comes from the relative Mayer-Vietoris sequence of $(M,\partial M) = (f^{-1}(V), C) \cup (\overline{M - f^{-1}(V)}, \partial M - C)$ and ∂_U comes from the relative Mayer-Vietoris sequence of $(U, U - int\, K) = (V, V) \cup (\overline{U-V}, \overline{U-(V\cup K)})$. A chase through the zig-zag lemma shows that ∂_M takes the orientation class of $(M,\partial M)$ to the class of $f^{-1}(\partial V)$ and ∂_U takes the generator of the infinite cyclic group $H_3(U, U - int\, K)$ to the class of ∂V. The proof is completed by noting that inclusion induces an isomorphism of $H_3(U, U - int\, K)$ with $H_3(U, U-K)$ (since $U - int\, K$ and $U - K$ both deformation retract onto U minus the interior of a regular neighborhood of K).

Note that $f|_C : C \to V$ and $f\,|_{f^{-1}(\partial V)}\colon f^{-1}(\partial V) \to V$ are homologous and hence $f|_C$ is homologous (in V) to a nonzero multiple of ∂V.

Applying the prime factorization theorem for compact 3-manifolds to a regular neighborhood of $f(C)$ in $int(V)$ we conclude the existence of an embedding R in $int(V)$ where R is a closed, oriented surface of genus zero such that R is homologous to $f|_C$ in $int(V)$ (recall that the prime factorization and the sphere theorem imply that π_2 of a compact 3-manifold is generated, as a π_1-module, by a finite family of pairwise disjoint embedded 2-spheres, and by the Hurewicz theorem H_2 is isomorphic to π_2 modulo the action of π_1). Now let W be a regular neighborhood of R in $int(V)$ and denote by $N(V)$ that component of $cl(V-W)$ containing ∂V. It remains only to show that $N(V)$ is compact. Suppose otherwise. Then there exists a proper ray in V extending from ∂V to infinity and avoiding R. Such a ray has intersection number one with ∂V but intersection number zero with R. This contradicts the fact that R is homologous in V to a multiple of ∂V. □

Definition 3.1 An *admissible pair* is a map $f : (X, Y) \to (M, M - L)$ and a subspace $K \subset L$ satisfying the following conditions:

1. (X, Y) is a pair of compact simplicial complexes and X is simply connected.

2. M is a compact orientable 3-manifold and K and L are compact 3-submanifolds with $K \subset int(L)$, $L \subset int(M)$ and L connected.

3. The map f is simplicial and M is an (abstract) regular neighborhood of $f(X)$.

4. Only one component of $X - f^{-1}(\partial L)$ has image under f meeting K.

5. The map $f_* : H_3(X, Y) \to H_3(M, M - L)$ is non-trivial.

We will refer to the M above as the *target* of the admissible pair.

Notation: If $X \subset P$ then ∂X denotes the frontier of X in P.

Lemma 3.2 *If $f : P \to U$ is the map of the theorem and K is a compact 3-submanifold of U then we can choose $X \subset P$ and a compact 3-submanifold L of U such that the pair $\{f|_X : (X, \partial X) \to (M, M - L), K\}$ is admissible where M is a regular neighborhood of $f(X)$ in U.*

Proof: Consider L_0 be a regular neighborhood of K in U. The hypothesis implies the existence of some X_0 with $f^{-1}(L_0) \subset X_0 \subset P$, such that $f_* : H_3(P, P - X_0) \to H_3(U, U - L_0)$ is non-trivial. Moreover once such an X_0 is chosen then larger X with $int(X) \supset X_0$ are also convenient for this purpose. Moreover f is proper implies the existence of an $X_1 \supset X_0$ such that $f(\partial X_1) \cap L_0 = \emptyset$. Using excision on both sides above we find that the map $(f|_{X_1})_* : H_3(X_1, \partial X_1) \to H_3(M, M - L_0)$ is non-trivial, where M denotes a regular neighborhood of $f(X_1)$ in U. Denote the set X_1 with these properties by $X(L_0)$. Note that $X(L_0)$ is defined for any compact L_0 engulfing K. If $X - f^{-1}(\partial L_0)$ has at least two components each of which has image meeting K then $f^{-1}(K) \cap (X(L_0) - f^{-1}(\partial L_0))$ is not connected. Since f is simplicial the latter is a simplicial subcomplex of P and so it cannot be path connected. In particular there exist points $x, y \in f^{-1}(K)$ such that any path connecting them in $X(L_0)$ should meet $f^{-1}(\partial L_0)$. But $f^{-1}(K)$ is a compact (thus finite) simplicial complex because f is proper hence it has a finite number of components. Consider some arcs joining these components in P. The union of these arcs with the components of $f^{-1}(K)$ is contained in some compact subset $K' \subset P$. Consider now L large enough such that $f^{-1}(\partial L) \cap K' = \emptyset$, and $K' \subset int(f^{-1}(L))$. Then $X(L) \supset f^{-1}(L) \supset K'$, and we claim that this X fulfills all conditions needed. If x, y are two points in $f^{-1}(K)$ then there exists a path connecting them inside K' and so there exists a path inside $X(L) - f^{-1}(\partial L)$. $\square$

Lemma 3.3 *Given an open, connected 3-manifold U and compactum X in U there exists a compact 3-submanifold K of U containing X such that each component of $cl(U - K)$ is noncompact and has connected boundary.*

Proof: Let M be a compact 3-submanifold of U containing X. Let N be the union of M and all compact components of $cl(U-M)$. To obtain K from N add 1-handles to N (in $cl(U-N)$) which connect different components of ∂N which are in the same component of $cl(U-N)$. $\square$

The proof of Proposition 2.1 will proceed by applying the tower construction to the admissible pair of Lemma 3.2 (where K is also chosen to satisfy the conclusion of Lemma 3.3) to obtain a map satisfying the hypothesis of Lemma 3.1. It is convenient to state first the following definition.

Definition 3.2 A *reduction of the admissible pair* $\{f_0 : (X,Y) \to (M_0, M_0 - L_0), K_0\}$ is a second admissible pair $\{f_1 : (X,Y) \to (M_1, M_1 - L_1), K_1\}$ such that there exists a map $p : M_1 \to M_0$, the "projection map" of the reduction, satisfying the following conditions:

1. $p \circ f_1 = f_0$.
2. $p(K_1) = K_0$, $p(L_1) = L_0$, and the maps $p|_{L_1} : L_1 \to L_0$, $p|_{K_1} : K_1 \to K_0$ are boundary preserving.
3. $p|_{L_1} : L_1 \to L_0$ has non-zero degree.
4. The complexity of f_1 is strictly less than the complexity of f_0 (where the complexity of a simplicial map g with compact domain is the number of simplexes s in the domain for which $g^{-1}(g(s)) \neq s$).
5. The image under p of only one component of $M_1 - \partial L_1$ meets K_0. Observe that, by (2), the image of L_1 must meet K_0.

Lemma 3.4 *If f_1 is a reduction of f_0 and f_2 is a reduction of f_1 then f_2 is a reduction of f_0.*

Proof: This is obvious.$\square$

Lemma 3.5 *Any admissible pair with non-simply connected target has a reduction with simply-connected target.*

Proof: It will suffice to show that any admissible pair with non-simply connected target has a reduction. Because then, by iteration (which could occur at most finitely many times by condition 4) and applying Lemma 3.3 we obtain a reduction with simply connected target.

Let $\{f|_X : (X,Y) \to (M, M-L), K\}$ be the admissible pair. Let $\widetilde{M}$ be the universal covering of M and p be the covering projection. Since X is simply connected there exists a lift $\widetilde{f} : X \to \widetilde{M}$ of f. We will show that $\{f_1 : (X,Y) \to (M_1, M_1 - L_1), K_1\}$ is a reduction, where

1. $f_1 = \widetilde{f}$.

2. M_1 is a regular neighborhood of $\widetilde{f}(X)$.

3. L_1 is the only component of $p^{-1}(L)$ which is contained in $f_1(X) = \widetilde{f}(X)$.

4. $K_1 = p^{-1}(K_0) \cap L_1$.

5. The covering map restricted to M_1 is the map p from the definition 3.2.

Let us show that f_1 is well-defined and admissible.

f_1 is well-defined: First we show that $f_1(Y) \subset M_1 - L_1$. We know that $f(Y) \subset M - L$ since f is admissible, and so $f_1(Y) = M_1 \cap p^{-1}(f(Y)) \subset M_1 \cap p^{-1}(M - L) = M_1 - p^{-1}(L) \subset M_1 - L_1$.

In order to have a consistent definition of L_1 we must show that there exists one and only one component of $p^{-1}(L)$ contained in $f_1(X)$.

First we prove the existence. Since f is non-trivial on H_3 (the condition (5) for f) we derive that $\widetilde{f}_* : H_3(X, Y) \to H_3(\widetilde{M}, \widetilde{M} - p^{-1}(L))$ is non-trivial. In particular the abelian group $H_3(\widetilde{M}, \widetilde{M} - p^{-1}(L))$ is non-zero. Since this group is freely generated by an equivariant regular neighborhood of the compact components of $p^{-1}(L)$, there exists at least one such (the deck transformations act transitively on the components of $p^{-1}(L)$ and so every component is compact). Since L is connected $H_3(M, M - L)$ is generated by the orientation class of a regular neighborhood of L and f_* nontrivial implies that L is in $f(X)$.

Observe now that $p(M_1 \cap p^{-1}(L)) = L$ since $p(M_1) = M$, and also $p(M_1 - M_1 \cap p^{-1}(L)) = M - L$. Then since f is H_3 nontrivial the map $\widetilde{f}_* : H_3(X, Y) \to H_3(M_1, M_1 - M_1 \cap p^{-1}(L))$ should be non-trivial. The same argument used above shows that $\widetilde{f}(X) \supset M_1 \cap p^{-1}(L)$. Let L_1 be a component of $p^{-1}(L)$ which meets $\widetilde{f}(X)$. Suppose that L_1 is not completely contained in $\widetilde{f}(X)$. Since L_1 is connected then $L_1 \cap (M_1 - \widetilde{f}(X)) \neq \emptyset$, or in other words the regular neighborhood of $\widetilde{f}(X)$ meets a larger subset of L_1 than the image $\widetilde{f}(X)$. This contradicts the fact that $M_1 \cap L_1 \subset \widetilde{f}(X)$. Thus any component of $p^{-1}(L)$ meeting $\widetilde{f}(X)$ is entirely contained in $\widetilde{f}(X)$. At least one component has non-void intersection with the image because $p(\widetilde{f}(X) \cap p^{-1}(L)) = p(M_1 \cap p^{-1}(L)) = L$.

Suppose now that there are two components, L_1 and L_1' meeting $\widetilde{f}(X)$. The two components are then disjoint and contained in $\widetilde{f}(X)$. Furthermore there are two components of $M_1 - p^{-1}(\partial L)$, namely $int(L_1)$ and $int(L_1')$ whose images under p meet K. However there is only one component, say ξ, of $X - f^{-1}(\partial L)$ whose image by f meets K. Then $\widetilde{f}(\xi) \subset M_1 - \partial L_1 \cup \partial L_1'$ since $\widetilde{f}(\xi)$ avoids $p^{-1}(\partial L)$ and $\widetilde{f}(\xi) \supset int(L_1) \cup int(L_1') \supset K_1 \cup K_1'$ because $p(f(\xi))$ meets K. This is a contradiction as $\widetilde{f}(\xi)$ must be connected.

f_1 is admissible: Conditions (1-3) from definition 3.1 are immediate. The condition (5) for f_1 is satisfied since $f = p \circ f_1$ is H_3-nontrivial. Finally (4) is implied by $f_1^{-1}(\partial L_1) =$

$f^{-1}(\partial L)$.

f_1 is a reduction of f: With the exception of (3) and (5) we leave them to the reader. Condition (3) follows from the fact that $p|_{L_1} : L_1 \to L$ is a covering map, and it is well-known that any covering map from one compact, connected, orientable 3-manifold to another has non-zero degree. An easy argument is the following. Triangulate and orient the base space. Lift the triangulation and the orientation of each 3-simplex to the covering space. If this does not give an orientation of the covering space, then there will be a 2-simplex which is a face of two 3-simplexes which are mapped to the same 3-simplex in the base space, contradicting the fact that the map is a covering map. It then follows that a fundamental cycle for the covering space is sent to n times the fundamental cycle for the base space, where n is the number of sheets of the covering.

We already saw before that $int(L_1)$ is the only one component of $M_1 - p^{-1}(\partial L)$ whose image meets K hence establishing (5). $\square$

Proof of Proposition 2.1 from the Lemmas: Begin with the admissible pair of Lemma 3.2 where K also satisfies the conclusion of Lemma 3.3. Apply Lemma 3.5 to that admissible pair. Note that the projection map of that reduction satisfies the hypothesis of Lemma 3.1 whose application then completes the proof. $\square$

4 Proof of Proposition 2.2

Lemma 4.1 *Suppose the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & X \\ h\downarrow & & \downarrow\psi \\ N & \subset & U \end{array}$$

is commutative and satisfies the following conditions:

1. *M and N are compact, connected, orientable 3-manifolds with $h(\partial M) \subset \partial N$.*

2. *h has non-zero degree.*

3. *X is simply-connected.*

Then $e_\sharp(\pi_1(N))$ is a torsion subgroup of $\pi_1(U)$ (where e denotes the inclusion $N \subset U$).

Proof: Let $p : \widetilde{N} \to N$ be the covering map such that $p_\sharp(\pi_1(\widetilde{N})) = h_\sharp(\pi_1(M))$. Then h lifts to $\tilde{h} : M \to \widetilde{N}$. Since the degree of h is non-zero $H_3(\widetilde{N}, \partial\widetilde{N}) \neq 0$, and so $\widetilde{N}$ is compact and thus p is finite sheeted. Hence $h_\sharp(\pi_1(M))$ has finite index in $\pi_1(N)$. (This is a standard argument.) Thus for any element of $\pi_1(N)$ some power will be in the image of $h_\sharp$ and therefore will be trivial in $\pi_1(U)$. $\square$

Now to prove Proposition 2.2 it will suffice to show that if N is a compact, connected 3-submanifold of U then $e_\sharp(\pi_1(N))$ is a torsion subgroup of $\pi_1(U)$. Let $f : X \to U$ be as in the definition of "H_3-semi-dominated" (where X is g.s.c.). By excision we have $f_* : H_3(Y, \partial Y) \to H_3(N, \partial N)$ is nontrivial, where $Y = f^{-1}(N)$. We can "realize" any element of $H_3(Y, \partial Y)$ by a map $g : (M, \partial M) \to (Y, \partial Y)$, where M is a compact orientable 3-manifold (i.e. $g_* : H_3(M, \partial M) \to H_3(Y, \partial Y)$ sends the orientation class of M to the preassigned element of $H_3(Y, \partial Y)$ - see [8]). The 3-manifold M coming from Thom's theorem might not be connected. But since $(f \circ g)_* : H_3(M, \partial M) \to H_3(N, \partial N)$ is nontrivial there will be some component M_0 of M such that $((f \circ g)|_{M_0})_* : H_3(M_0, \partial M_0) \to H_3(N, \partial N)$ is nontrivial. Now apply the lemma with $h = f \circ (g|_{M_0})$. $\square$

5 Proof of Proposition 2.3

Lemma 5.1 *If A and B are groups and $a \in A$, $b \in B$ are neither the identity then $a * b$ is not a torsion element in $A * B$ (the free product of A and B).*

Proof: This is a standard fact from combinatorial group theory. Every element of finite order in $A * B$ is conjugate to an element of A or of B (Corollary 4.1.4 of [3]), and every element of $A * B$ is conjugate to a cyclically reduced word which is unique up to cyclic permutation (Theorem 4.2 of [3]). Since $a * b$ is cyclically reduced it cannot be conjugate to an element of A or of B and hence cannot have finite order. $\square$

Corollary 5.1 *The prime factorization of a closed 3-manifold whose fundamental group is a torsion group can have at most one non-simply connected factor.*

Now to prove Proposition 2.3 we assume that U is not simply-connected (otherwise we are done) and let M be a non-simply connected 3-submanifold of U such that ∂M has genus zero. We can express M as a connect-sum of a punctured homotopy 3-ball and a closed orientable 3-manifold N where $\pi_1(N)$ is a torsion group. By the above corollary we can assume N is irreducible. By [1] any orientable, irreducible closed 3-manifold with torsion must have finite fundamental group. It remains only to show that U can have no other non-simply connected factor but this also follows from the Corollary. $\square$

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